

# SOME ESTIMATES FOR IMAGINARY POWERS OF LAPLACE OPERATORS IN VARIABLE LEBESGUE SPACES AND APPLICATIONS

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ABSTRACT. In this paper we study some estimates of norms in variable exponent Lebesgue spaces for a singular integral operators that are imaginary powers of the Laplace operator in  $\mathbb{R}^n$ . Using Mellin transform argument, from this estimates we obtain boundedness for a family of maximal operators in variable exponent Lebesgue spaces, which are closely related to the (weak) solution of the wave equation.

## 1. INTRODUCTION

In the recent paper [10] we study boundedness of Stein's spherical maximal function  $\mathcal{M}$  in variable exponent Lebesgue spaces. The proof is based on the Rubio De Francia extrapolation methods and corresponding results in weighted Lebesgue spaces. The Stein's spherical maximal functions are closely related to the solution of the wave equation in  $\mathbb{R}^3$ . To study the wave equation in  $\mathbb{R}^n$ ,  $n > 3$  we need to consider more general spherical maximal function  $\mathcal{M}^\alpha$ , when  $\alpha = \frac{3-n}{2}$ . To investigate such operator we used new approach based on Mellin transform arguments used first time by Cowling and Mouceri in [3]. which reduced the problem to find the sharp estimates for norms of imaginary power of Laplace operator.

We define  $\mathcal{P}(\mathbb{R}^n)$  to be the set of all measurable functions  $p : \mathbb{R}^n \rightarrow [0, \infty]$ . Functions  $p \in \mathcal{P}(\mathbb{R}^n)$  are called variable exponents on  $\mathbb{R}^n$ . We define  $p^- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x)$  and  $p^+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)$ . If  $p^+ < \infty$ , then we call  $p$  a bounded variable exponent.

If  $p \in \mathcal{P}(\mathbb{R}^n)$ , then we define  $p' \in \mathcal{P}(\mathbb{R}^n)$  by  $\frac{1}{p(y)} + \frac{1}{p'(y)} = 1$ , where  $\frac{1}{\infty} := 0$ . The function  $p'$  is called the dual variable exponent of  $p$ .

Let  $p \in \mathcal{P}(\mathbb{R}^n)$ ,  $L^{p(\cdot)}(\mathbb{R}^n)$  denotes the set of measurable functions  $f$  on  $\mathbb{R}^n$  such that for some  $\lambda > 0$

$$\int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

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This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Let  $B(x, r)$  denote the open ball in  $\mathbb{R}^n$  of radius  $r$  and center  $x$ . By  $|B(x, r)|$  we denote  $n$ -dimensional Lebesgue measure of  $B(x, r)$ . The Hardy-Littlewood maximal operator  $M$  is defined on locally integrable function  $f$  on  $\mathbb{R}^n$  by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

Define the Spherical Maximal operator  $\mathcal{M}$ , by

$$\mathcal{M}f(x) := \sup_{t>0} |\mu_t * f(x)| = \sup_{t>0} \left| \int_{\{y \in \mathbb{R}^n : |y|=1\}} f(x - ty) d\mu_1(y) \right|$$

where  $\mu_t$  denotes the normalized surface measure on the sphere of center 0 and radius  $t$  in  $\mathbb{R}^n$ . The Hardy-Littlewood maximal operator  $M$ , which involves averaging over balls, is clearly related to the spherical maximal operator, which averages over spheres. Indeed, by using polar coordinates, one easily verifies the pointwise inequality  $Mf(x) \leq \mathcal{M}f(x)$  for any (continuous) function.

We say that a function  $p : \mathbb{R}^n \rightarrow (0, \infty)$  is locally log-Hölder continuous on  $\mathbb{R}^n$  if there exists  $c_1 > 0$  such that

$$|p(x) - p(y)| \leq c_1 \frac{1}{\log(e + 1/|x - y|)}$$

for all  $x, y \in \mathbb{R}^n$ . We say that  $p(\cdot)$  satisfies the log-Hölder decay condition if there exist  $p_\infty \in (0, \infty)$  and a constant  $c_2 > 0$  such that

$$|p(x) - p_\infty| \leq c_2 \frac{1}{\log(e + |x|)}$$

for all  $x \in \mathbb{R}^n$ . We say that  $p(\cdot)$  is globally log-Hölder continuous on  $\mathbb{R}^n$  if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

We say  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$  if  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $\frac{1}{p(\cdot)}$  is globally log-Hölder continuous on  $\mathbb{R}^n$ . If  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  with  $p^+ < \inf ty$ , then  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$  if and only if  $p$  is globally log-Hölder continuous on  $\mathbb{R}^n$ .

If  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$  and  $p^- > 1$ , then the classical boundedness theorem for the Hardy-Littlewood maximal operator can be extended to  $L^{p(\cdot)}$  (see [6, 7, 5, 8]). If  $n \geq 3$ ,  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$  and  $\frac{n}{n-1} < p^- \leq p^+ < p^-(n-1)$ , then the boundedness theorem for spherical maximal function  $\mathcal{M}$  in  $L^{p(\cdot)}$  was proved in [10].

Denote by  $\mathcal{B}(\mathbb{R}^n)$  the class all measurable functions  $p : \mathbb{R}^n \rightarrow (0, \infty)$  for which the Hardy-Littlewood maximal operator is bounded on  $L^{p(\cdot)}$ .

## 2. IMAGINARY POWER OF LAPLACE OPERATOR IN VARIABLE LEBESGUE SPACES

Let  $S(\mathbb{R}^n)$  denote the Schwartz space, consisting of sufficiently smooth functions that are rapidly decreasing at infinity. Let  $\Delta$  be the standard Laplace operator in  $\mathbb{R}^n$ , given by

$$\Delta = \sum_{j=1}^n \partial_j^2.$$

If  $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi x \cdot \xi} f(x) dx$ , then

$$(-\Delta f)^\wedge(\xi) = |2\pi\xi|^2 \hat{f}(\xi), \quad f \in S(\mathbb{R}^n)$$

Inspired by this relation, one may define  $\Delta^{\beta/2}$  for any complex exponent  $\beta$  by

$$(-\Delta^{\beta/2})^\wedge(\xi) = (2\pi|\xi|)^\beta \hat{f}(\xi), \quad f \in S(\mathbb{R}^n).$$

In particular, for each  $0 < \alpha < n$ , the operator

$$I_\alpha : f \mapsto \Delta^{\alpha/2} f$$

is known as the Riesz potential. Here  $I_\alpha$  may be expressed as

$$I_\alpha f = K_\alpha * f$$

where  $K_\alpha(x) = \pi^{-n/2} 2^{-\alpha} \Gamma\left(\frac{n-\alpha}{2}\right) / \Gamma\left(\frac{\alpha}{2}\right) |x|^{-n+\alpha}$ , and this tells us that  $I_\alpha$  is an integral operator (see [19] p. 117).

In this paper we shall consider the operator  $I_{iu}$ ,  $u \in \mathbb{R} \setminus \{0\}$ , given by

$$I_{iu} f = K_{iu} * f, \quad f \in S(\mathbb{R}^n),$$

which makes sense via

$$(I_{iu} f)^\wedge(\xi) = (2\pi|\xi|)^{-iu} \hat{f}(\xi), \quad f \in S(\mathbb{R}^n),$$

that is  $I_{iu} = (-\Delta)^{-iu}$  an imaginary power of  $\Delta$ . This operator was studied by Muckenhoupt [17] in 1960 and used by Cowling and Mauceri [3] in 1978 to prove Stein's theorem on the spherical maximal function.

Note that  $|\hat{K}(\xi)| = |(2\pi|\xi|)^{-iu}| = 1$ , so that by Plancherel's theorem we have in  $L^2(\mathbb{R}^n)$

$$(2.1) \quad \|I_{iu} f\|_2 = \|f\|_2.$$

By using further properties of the kernel  $K_{iu}$ , particularly the fact that it is locally integrable away from the origin and satisfies

$$|K_{iu}(x)| \leq C(1 + |u|)^{n/2} |x|^{-n}$$

and

$$|\nabla K_{iu}(x)| \leq C(1 + |u|)^{n/2+1} |x|^{-n-1}$$

for  $x \neq 0$ , one may observe that  $I_{iu}$  also extends to a bounded operator on  $L_w^p(\mathbb{R}^n)$ .

Under a weight we mean a non-negative, locally integrable function  $w$ . When  $1 < p < \infty$ , we say  $w \in A_p$  if for every ball  $Q$

$$\frac{1}{|Q|} \int_Q w(x) dx \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} \leq C < \infty.$$

By  $A_{p,w}$  we denote the infimum over the constants on the right-hand side of the last inequality.

**Theorem 2.1** ([13]). *Let  $1 < p < \infty$  and  $w \in A_p$ . For each  $\delta \in (0, 1)$  and  $u \in \mathbb{R} \setminus \{0\}$ , the following weighted estimate holds whenever  $w \in A_p$ ,  $1 < p < \infty$ :*

$$(2.2) \quad \|I_u f\|_{p,w} \leq C(1 + |u|)^{n/2+\delta} \|f\|_{p,w}, \quad f \in L_w^p(\mathbb{R}^n).$$

Our result in variable Lebesgue spaces are following

**Theorem 2.2.** *Let  $p(\cdot) \in \mathcal{B}^{\log}(\mathbb{R}^n)$ . Then for all  $\delta \in (0, 1)$  there exists a constant  $C$  such that, for all  $u \in \mathbb{R} \setminus \{0\}$*

$$(2.3) \quad \|I_{iu} f\|_{p(\cdot)} \leq C(1 + |u|)^{n/2+\delta} \|f\|_{p(\cdot)}, \quad f \in L^{p(\cdot)}(\mathbb{R}^n).$$

To prove this Theorem we need the extrapolation theorem for variable Lebesgue spaces. By  $\mathcal{F}$  we will denote a family of ordered pairs of non-negative, measurable functions  $(f, g)$ . We say that an inequality

$$(2.4) \quad \int_{\mathbb{R}^n} f(x)^{p_0} w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w(x) dx, \quad (0 < p_0 < \infty)$$

holds for any  $(f, g) \in \mathcal{F}$  and  $w \in A_q$  (for some  $q, 1 < q < \infty$ ) if it holds for any pair in  $\mathcal{F}$  such that the left-hand side is finite, and the constant  $C$  depends only on  $p_0$  and on the constant  $A_{q,w}$ .

**Theorem 2.3.** ([8, Theorem 7.2.1, page 214]). *Given a family  $\mathcal{F}$ , assume that (2.4) holds for some  $1 < p_0 < \infty$ , for every weight  $w \in A_{p_0}$  and for all  $(f, g) \in \mathcal{F}$ . Let exponent  $p(\cdot)$  be such that there exists  $1 < p_1 < p_-$ , with  $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$ . Then*

$$\|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}$$

for all  $(f, g) \in \mathcal{F}$  such that  $f \in L^{p(\cdot)}(\mathbb{R}^n)$ .

*Proof of the Theorem 2.2.* Using Theorem 2.3, estimate (2.2) and the fact that if  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  then  $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$  for some  $1 < p_1 < p_-$ , (see [8, Theorem 5.7.2, page 181]) we obtain (2.3).  $\square$

**Corollary 2.4.** *Let  $\frac{1}{p(\cdot)} = \frac{1-\theta}{2} + \frac{\theta}{\tilde{p}(\cdot)}$  for some  $0 < \theta < 1$  and  $\tilde{p}(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Then for all  $0 < \delta < 1$  there exists a constant  $C$  such that, for all  $u \in \mathbb{R} \setminus \{0\}$*

$$(2.5) \quad \|I_{iu}f\|_{p(\cdot)} \leq C(1 + |u|)^{\theta n/2 + \theta\delta} \|f\|_{p(\cdot)}, \quad f \in L^{p(\cdot)}(\mathbb{R}^n).$$

*Proof.* By using the complex interpolation theorem for variable exponent Lebesgue spaces (see [8, Theorem.1.2, page 215]), we have  $L^{p(\cdot)}(\mathbb{R}^n) = [L^2(\mathbb{R}^n), L^{\tilde{p}(\cdot)}(\mathbb{R}^n)]_\theta$ . Therefore,  $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  and

$$\|I_{iu}\|_{L^{p(\cdot)} \rightarrow L^{p(\cdot)}} \leq \|I_u\|_{L^2 \rightarrow L^2}^{1-\theta} \|I_u\|_{L^{\tilde{p}(\cdot)} \rightarrow L^{\tilde{p}(\cdot)}}^\theta \leq C(1 + |u|)^{\theta n/2 + \theta\delta}.$$

$\square$

### 3. SPHERICAL MAXIMAL FUNCTION

For  $\alpha > 0$ , let  $m_\alpha(x) = (1 - |x|^2)^{\alpha-1}/\Gamma(\alpha)$ , where  $|x| < 1$ , and  $m_\alpha(x) = 0$  if  $|x| \geq 1$ . With  $m_{\alpha,t}(x) = m_\alpha(x/t)t^{-n}$ ,  $t > 0$ , we define spherical means of (complex) order  $\operatorname{Re} \alpha > 0$ , by

$$\mathcal{M}_t^\alpha f(x) = (m_{\alpha,t} * f)(x).$$

Note that the Fourier transform of  $m_\alpha$  is given by

$$\widehat{m}_\alpha(\xi) = \pi^{-\alpha+1} |\xi|^{-n/2-\alpha+1} J_{n/2+\alpha}(2\pi|\xi|).$$

The definition of  $\mathcal{M}_t^\alpha$  can be extended to the region  $\operatorname{Re} \alpha \leq 0$  by the analytic continuation. Indeed for complex  $\alpha$  in general we can define the operator  $\mathcal{M}_t^\alpha$  by

$$(\mathcal{M}_t^\alpha f)^\wedge(\xi) = \widehat{m}_\alpha(t\xi) \widehat{f}(\xi), \quad f \in C_0^\infty(\mathbb{R}^n).$$

Define the spherical maximal operator of order  $\alpha$  by

$$\mathcal{M}^\alpha f(x) = \sup_{t>0} |\mathcal{M}_t^\alpha f(x)|.$$

we observe that for  $\alpha = 0$  we have  $\mathcal{M}^\alpha f(x) = c\mathcal{M}f(x)$  for appropriate constant  $c$ .

**Theorem 3.1** (Stein). *The inequality  $\|\mathcal{M}^\alpha f\|_p \leq A_{p,\alpha}\|f\|_p$  holds in the following circumstances:*

- (a) if  $1 < p \leq 2$ , when  $\alpha > 1 - n + n/p$ .
- (b) if  $2 \leq p \leq \infty$ , when  $\alpha > (1/p)(2 - n)$ .

Note that for  $\alpha$  in Steins's theorem we have restriction  $1 - n/2 < \alpha < 1$  and also we have

- a) if  $\alpha = 0$ , then  $n \geq 3$ ,  $p > \frac{n}{n-1}$ .
- b) If  $0 < \alpha < 1$  then  $n \geq 2$ ,  $\frac{n}{n-1+\alpha} < p \leq \infty$ .
- c) If  $1 - n/2 < \alpha < 0$  then  $n \geq 3$ ,  $\frac{n}{n-1+\alpha} < p < \frac{n-2}{-\alpha}$ .

In this section we study boundedness properties of the Stein's spherical maximal operator  $\mathcal{M}^\alpha$  on the variable exponent Lebesgue spaces. Our main result is following

**Theorem 3.2.** *Let for some  $\varepsilon$ , where  $0 < \varepsilon < 1 - \frac{2}{n} + \frac{2}{n}\alpha$  and  $\tilde{p}(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , If we have  $L^{p(\cdot)}(\mathbb{R}^n) = [L^2(\mathbb{R}^n), L^{\tilde{p}(\cdot)}(\mathbb{R}^n)]_\varepsilon$ . then spherical maximal operator  $\mathcal{M}^\alpha$  bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ .*

*Proof.* Let  $F_\alpha(\lambda) = c\lambda^{-n/2-\alpha+1}J_{n/2+\alpha-1}(2\pi\lambda)$ ,  $\lambda > 0$ . Then we have

$$(M_t^\alpha f)^\wedge(\xi) = F_\alpha(t|\xi|)\hat{f}(\xi)$$

Let  $F_\alpha^*(\lambda) = F_\alpha(\lambda) - F_\alpha(0)e^{-\lambda^2}$ . So that we have  $F_\alpha^*(0) = 0$ . Using the Mellin transform we have

$$F_\alpha^*(\lambda) = \int_{\mathbb{R}} A_\alpha(u)\lambda^{iu}du, \quad \lambda > 0,$$

that is,  $F_\alpha^*$  is the Mellin transform of  $A_\alpha(u)$  for Mellin transform see[11]. By the Fourier Inversion Theorem, this holds if and only if

$$\begin{aligned} A_\alpha(u) &= \frac{1}{2\pi} \int_0^\infty F_\alpha^*(\lambda)\lambda^{-1-iu}d\lambda, \quad u \in \mathbb{R}. \\ A_\alpha(u) &= \frac{\Gamma(\alpha)\Gamma(-\frac{iu}{2})}{4\pi^{1/2}}. \end{aligned}$$

For  $f \in \mathcal{S}(\mathbb{R}^n)$  we have

$$\begin{aligned} m_t^\alpha * f(x) &= (\widehat{m_t^\alpha}(\xi) \cdot \hat{f}(\xi))^\vee = (F_\alpha(t|\xi|) \cdot \hat{f})^\vee(x) \\ &= ((F_\alpha^*(t|\xi|) + ce^{-|t\xi|^2}) \cdot \hat{f}(\xi))^\vee(x) \end{aligned}$$

We have

$$\begin{aligned} (F_\alpha^*(t|\xi|) \cdot \hat{f}(\xi))^\vee(x) &= \left( \int_{\mathbb{R}} A_\alpha(u)(t|\xi|)^{iu}du \hat{f}(\xi) \right)^\vee(x) \\ &= \int_{\mathbb{R}} A_\alpha(u)t^{iu}(|\xi|)^{iu} \hat{f}(\xi)^\vee(x)du \\ &= \int_{\mathbb{R}} A_\alpha(u)t^{iu}(|\cdot|^{-n+iu} * f)(x)du \end{aligned}$$

we need estimate of  $A_\alpha(u)$ ,  $u \rightarrow \infty$ .

For  $A_\alpha(u)$  we have

$$A_\alpha(u) = \frac{\Gamma(\alpha)\Gamma(-\frac{iu}{2})}{4\pi^{1/2}} \left[ \frac{2^{-iu}}{\Gamma(\alpha + 1/2 + iu/2)} - \frac{1}{\Gamma(\alpha + 1/2)} \right]$$

and we have  $A_\alpha(u) = O((1 + |u|)^{-\operatorname{Re} \alpha - 1/2})$ .

Using Minkovski's inequality we obtain

$$\begin{aligned}
\|(F_\alpha^*(t|\xi|) \cdot \widehat{f}(\xi))^\vee(x)\|_{p(\cdot)} &= \left\| \int_R A_\alpha(u) t^{iu} (|\cdot|^{-n+iu} * f)(x) du \right\|_{p(\cdot)} \\
&\leq \int_R \|A_\alpha(u) t^{iu} (|\cdot|^{-n+iu} * f)(x)\|_{p(\cdot)} du \\
&= \int_R A_\alpha(u) \|(|\cdot|^{-n+iu} * f)(x)\|_{p(\cdot)} du \\
&\leq C \|f\|_{p(\cdot)}.
\end{aligned}$$

□

To obtain more particular results we need the following

**Lemma 3.3.** *Let  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ . If  $1 - n/2 < \alpha < 1$  and  $\frac{n}{n-1+\alpha} < p_- \leq p_+ < \frac{n}{1-\alpha}$ , then there are exist  $\varepsilon$ ,  $0 < \varepsilon < 1 - \frac{2}{n} + \frac{2}{n}\alpha$  and variable exponent  $\widetilde{p}(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ , such that we have*

$$(3.1) \quad \frac{1}{p(\cdot)} = \frac{1-\varepsilon}{2} + \frac{\varepsilon}{\widetilde{p}(\cdot)}.$$

*Proof.* We need to find  $\varepsilon$  such that  $0 < \varepsilon < 1 - \frac{2}{n} + \frac{2}{n}\alpha$  and exponent  $\widetilde{p}(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$  such that (3.1) holds.

We have

$$\frac{1}{n} - \frac{\alpha}{n} < \inf_{x \in \mathbb{R}^n} \frac{1}{p(x)} \leq \sup_{x \in \mathbb{R}^n} \frac{1}{p(x)} < 1 - \frac{1}{n} + \frac{\alpha}{n}.$$

Let  $\frac{1}{p(x)} = \frac{1}{2} + r(x)$ . It is easy to see that

$$(3.2) \quad \frac{1}{n} - \frac{\alpha}{n} - \frac{1}{2} < \inf_{x \in \mathbb{R}^n} r(x) \leq \sup_{x \in \mathbb{R}^n} r(x) < \frac{1}{2} - \frac{1}{n} + \frac{\alpha}{n}.$$

Equation (3.1) is equivalent to

$$(3.3) \quad \frac{1}{2} + \frac{r(x)}{\varepsilon} = \frac{1}{\widetilde{p}(x)}.$$

Using (3.2) we may take small  $\delta > 0$  such that

$$\frac{1}{n} - \frac{\alpha}{n} - \frac{1}{2} + \delta < \inf_{x \in \mathbb{R}^n} r(x) \leq \sup_{x \in \mathbb{R}^n} r(x) < \frac{1}{2} - \frac{1}{n} + \frac{\alpha}{n} - \delta.$$

Then for  $\varepsilon$ ,  $0 < \varepsilon < 1 - \frac{2}{n} + \frac{2}{n}\alpha$ ,  $\varepsilon = 1 - \frac{2}{n} + \frac{2}{n}\alpha - \varepsilon_0$ ,  $\varepsilon_0 > 0$  we have

$$\frac{\frac{1}{n} - \frac{\alpha}{n} - \frac{1}{2} + \delta}{1 - \frac{2}{n} + \frac{2}{n}\alpha - \varepsilon_0} < \inf_{x \in \mathbb{R}^n} \frac{r(x)}{\varepsilon} \leq \sup_{x \in \mathbb{R}^n} \frac{r(x)}{\varepsilon} < \frac{\frac{1}{2} - \frac{1}{n} + \frac{\alpha}{n} - \delta}{1 - \frac{2}{n} + \frac{2}{n}\alpha - \varepsilon_0}$$

if we take  $\varepsilon_0 < 2\delta$  we get

$$(3.4) \quad -\frac{1}{2} < \inf_{x \in \mathbb{R}^n} \frac{r(x)}{\varepsilon} \leq \sup_{x \in \mathbb{R}^n} \frac{r(x)}{\varepsilon} < \frac{1}{2}$$

From (3.3) and (3.4) follows

$$0 < \inf_{x \in \mathbb{R}^n} \frac{1}{\widetilde{p}(x)} \leq \sup_{x \in \mathbb{R}^n} \frac{1}{\widetilde{p}(x)} < 1.$$

It is not hard to proof that  $\widetilde{p}(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ . Indeed we may use the simple equality

$$\left| \frac{1}{\widetilde{p}(x)} - \frac{1}{\widetilde{p}(y)} \right| = \varepsilon \left| \frac{1}{\widetilde{p}(x)} - \frac{1}{\widetilde{p}(y)} \right|.$$

□

*Remark 3.4.* In fact in the Lemma 3.3 we using only simple property that if  $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$  follows that also  $\tilde{p}(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ . For applications we need to consider such classes of exponents with this property, that Hardy-Littlewood maximal function are bounded on variable exponent Lebesgue spaces. In general the class of variable exponents  $\mathcal{B}(\mathbb{R})$ , for which the Hardy-Littlewood maximal function are bounded on variable exponent Lebesgue spaces have not such property (see [15]). The opposite statement by interpolation theorem is always true, that is if  $\tilde{p}(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , then  $p \in \mathcal{B}(\mathbb{R}^n)$ . For example (see [8, Remark 4.3.10, page 114]), it is possible to replace the log-Hölder decay condition by the weaker condition  $1 \in L^{s(\cdot)}(\mathbb{R}^n)$  with

$$\frac{1}{s(x)} := \left| \frac{1}{p(x)} - \frac{1}{p_\infty} \right|.$$

**Corollary 3.5.** *Let  $n \geq 3$ ,  $1 - n/2 < \alpha < 1$ ,  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$  and  $\frac{n}{n-1+\alpha} < p_- \leq p_+ < \frac{n}{1-\alpha}$ . Then spherical maximal functions  $\mathcal{M}^\alpha$  are bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ .*

*Proof.* Using Lemma 3.3 and complex interpolation method for variable exponent Lebesgue spaces from Theorem 3.2 we deduce a desired result. □

**Theorem 3.6.** *Let  $n \geq 2$  and  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$  and  $\frac{n}{n-1+\alpha} < p_- \leq p_+ < p_- \frac{n-1+\alpha}{1-\alpha}$ , ( $0 < \alpha < 1$ ). Then spherical maximal operator  $\mathcal{M}^\alpha$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ .*

*Proof.* Let fix  $\gamma$  such that  $1 < \gamma < \frac{p_-(n-1+\alpha)}{n}$  and  $p_+ \gamma < \frac{p_-(n-1+\alpha)}{1-\alpha}$ . Define the following variable exponent

$$\bar{p}(x) := \frac{p(x)n\gamma}{p_-(n-1+\alpha)}.$$

It is clear that  $\bar{p} \in \mathcal{P}^{\log}(\mathbb{R}^n)$ ,

$$\bar{p}_- = \frac{n\gamma}{n-1+\alpha} > \frac{n}{n-1+\alpha}$$

and

$$\bar{p}_+ = \frac{p_+n\gamma}{p_-(n-1+\alpha)} < \frac{n}{n-\alpha}.$$

By Lemma 3.3 there are exist  $\varepsilon$ ,  $0 < \varepsilon < 1 - \frac{2}{n} + \frac{2}{n}\alpha$  and variable exponent  $\tilde{p} \in \mathcal{P}^{\log}(\mathbb{R}^n)$ , such that

$$\frac{1}{\bar{p}(\cdot)} = \frac{1-\varepsilon}{2} + \frac{\varepsilon}{\tilde{p}(\cdot)}.$$

Therefore, by complex interpolation theorem for variable exponent Lebesgue spaces, we have

$$L^{\bar{p}(\cdot)}(\mathbb{R}^n) = [L^2(\mathbb{R}^n), L^{\tilde{p}(\cdot)}(\mathbb{R}^n)]_\varepsilon.$$

Now use Theorem 3.2 we deduce that spherical maximal functions  $\mathcal{M}^\alpha$  are bounded on  $L^{\bar{p}(\cdot)}(\mathbb{R}^n)$ .

By complex interpolation method for variable exponent Lebesgue spaces, we have

$$[L^\infty(\mathbb{R}^n), L^{\bar{p}(\cdot)}(\mathbb{R}^n)]_\theta = L^{p(\cdot)}(\mathbb{R}^n)$$

for  $\theta = \frac{n\gamma}{p_-(n-1+\alpha)} \in (0, 1)$ . By using the fact that if  $0 < \alpha < 1$  the spherical maximal functions  $\mathcal{M}^\alpha$  are bounded on  $L^\infty(\mathbb{R}^n)$  we obtain that spherical maximal functions  $\mathcal{M}^\alpha$  are bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ .  $\square$

**Corollary 3.7.** *Let  $n \geq 3$ ,  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$  and  $\frac{n}{n-1} < p_- \leq p_+ < p_-(n-1)$ . Then spherical maximal function  $\mathcal{M}$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ .*

*Proof.* If we take  $\alpha = 0$  in Theorem 3.6 we obtain that spherical maximal function  $\mathcal{M}$  is bounded on  $L^{p(\cdot)}(\mathbb{R}^n)$ .  $\square$

#### 4. AN APPLICATION

Spherical averages often make their appearance as solutions of partial differential equations.

a) Let  $\alpha = \frac{3-n}{2}$ . For an appropriate constant  $c_n$ , we have that  $u(x, t) = c_n t \mathcal{M}_t^\alpha(x)$ , where  $u$  is the solution of the wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x, t) &= \Delta_x(u)(x, t), \\ u(x, 0) &= 0, \\ \frac{\partial u}{\partial t}(x, t) &= f(x), \end{aligned}$$

(see [19], page 519).

**Corollary 4.1.** *Let  $n \geq 3$  and  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ . If  $\frac{2n}{n+1} < p_- \leq p_+ < \frac{2n}{n-1}$ , then for the solution  $u = u(x, t)$  of the wave equation with the initial data in  $f \in L^{p(\cdot)}(\mathbb{R}^n)$ , we have the following a priori estimate:*

$$\left\| \sup_{t>0} \frac{|u(x, t)|}{t} \right\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)},$$

where  $C$  depends on  $p(\cdot)$  only.

Also,  $\lim_{t \rightarrow 0} \frac{|u(x, t)|}{t} = f(x)$  a.e.  $x \in \mathbb{R}^n$ .

b) Let  $\alpha = \frac{3-n}{2}$ . For an appropriate constant  $c_n$ , the spherical average  $u(x, t) = c_n \mathcal{M}_t^\alpha(x)$  solves Darboux's equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x, t) + \frac{2}{t} \frac{\partial u}{\partial t}(x, t) &= \Delta_x(u)(x, t), \\ u(x, 0) &= f(x), \\ \frac{\partial u}{\partial t}(x, t) &= 0. \end{aligned}$$

**Corollary 4.2.** *Let  $n \geq 3$  and  $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ . If  $\frac{2n}{n+1} < p_- \leq p_+ < \frac{2n}{n-1}$ , then for the solution  $u = u(x, t)$  of the Darboux's equation with the initial data  $f \in L^{p(\cdot)}(\mathbb{R}^n)$  we have the following a priori estimate:*

$$\left\| \sup_{t>0} u(x, t) \right\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)},$$

where  $C$  depends on  $p(\cdot)$  only.

Also,  $\lim_{t \rightarrow 0} |u(x, t)|t = f(x)$  a.e.  $x \in \mathbb{R}^n$ .

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